SOME RELATIONAL STRUCTURES WITH POLYNOMIAL GROWTH AND THEIR ASSOCIATED ALGEBRAS

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This paper is dedicated to Adriano Garsia at the occasion of his 75th birthday

Abstract. The *profile* of a relational structure R is the function φ_R which counts for every integer n the number, possibly infinite, $\varphi_R(n)$ of substructures of R induced on the n-element subsets, isomorphic substructures being identified. Several graded algebras can be associated with R in such a way that the profile of R is simply the Hilbert function. An example of such graded algebra is the *age algebra* $\mathbb{K}.\mathcal{A}(R)$, introduced by P. J. Cameron. In this paper, we give a closer look at this association, particularly when the relational structure R decomposes into finitely many monomorphic components. In this case, several well-studied graded commutative algebras (e.g. the invariant ring of a finite permutation group, the ring of quasi-symmetric polynomials) are isomorphic to some $\mathbb{K}.\mathcal{A}(R)$. Also, φ_R is a quasi-polynomial, this supporting the conjecture that, with mild assumptions on R, φ_R is a quasi-polynomial when it is bounded by some polynomial.

Keywords: Relational structure, profile, graded algebra, Hilbert function, Hilbert series, polynomial growth, invariant ring, permutation group.

1. Presentation

A *relational structure* is a realization of a language whose non-logical symbols are predicates, that is a pair $R := (E, (\rho_i)_{i\in I})$ made of a set E and of a family of n_i-ary relations ρ_i on E. The family $\mu := (n_i)_{i\in I}$ is the *signature* of R. The *profile* of R is the function φ_R which counts for every integer n the number $\varphi_R(n)$ of substructures of R induced on the n-element subsets, isomorphic substructures being identified. Clearly, this function only depends upon the set $\mathcal{A}(R)$ of finite substructures of R considered up to an isomorphism, a set introduced by R. Fraïssé under the name of *age* of R (see [Fra00]). If I is finite $\varphi_R(n)$ is necessarily finite. As we will see, in order to capture examples coming from algebra and group theory, we cannot preclude I to be infinite. Since the profile is finite in these examples, we will always make the assumption that $\varphi_R(n)$ is finite, no matter how large I is.

A basic result about the behavior of the profile is this:

Theorem 1.1. *If R is a relational structure on an infinite set, then φ_R is non-decreasing.*

This result was obtained in 1971 by the first author (see Exercise 8 p. 113 [Fra71]). A proof based on linear algebra is given in [Pou76].

Provided that the relational structures satisfies some mild conditions, there are jumps in the behavior of the profile:

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Theorem 1.2. [Pou78] *Let $R := (E, (\rho_i)_{i\in I})$ be a relational structure. The growth of φ_R is either polynomial or as fast as every polynomial provided that either the signature $\mu := (n_i)_{i\in I}$ is bounded or the kernel $K(R)$ of R is finite.*

Note that a map $\varphi : \mathbb{N} \to \mathbb{N}$ has *polynomial growth*, of *degree k*, if $an^k \leq \varphi(n) \leq bn^k$ for some $a, b > 0$ and n large enough. The *kernel* of R is the set $K(R)$ of $x \in E$ such that $\mathcal{A}(R_{|E\setminus\{x\}}) \neq \mathcal{A}(R)$. Relations with empty kernel are the *inexhaustible* relations of R. Fraïssé (see [Fra00]). We call *almost inexhaustible* those with finite kernel. The hypothesis about the kernel is not ad hoc. As it turns out, if the growth of the profile of a relational structure with a bounded signature is bounded by a polynomial then its kernel is finite. Some hypotheses on R are needed, indeed *for every increasing and unbounded map $\varphi : \mathbb{N} \to \mathbb{N}$, there is a relational structure R such that φ_R is unbounded and eventually bounded above by φ*(cf. [Pou81]).

The consideration of examples suggested that in order to study the profile of a relational structure R, the right object to consider is rather the generating series

$$\mathcal{H}_{\varphi_R} := \sum_{n=0}^{\infty} \varphi_R(n) Z^n$$

This innocuous change leads immediately to several:

Questions 1.3.
1. For which relational structures is the series $\mathcal{H}_{\varphi_R}$ a rational fraction? A rational fraction of the form
$$\frac{P(Z)}{(1-Z)(1-Z^{n_2})\cdots(1-Z^{n_k})},$$
with $k \geq 1$, $1 = n_1 \leq n_2 \leq \cdots \leq n_k$, $P(0) = 1$, and $P \in \mathbb{Z}[Z]$?
2. Is this the case of relational structures with bounded signature or finite kernel for which the profile is bounded by some polynomial? When can P be taken with nonnegative coefficients?
3. For which relational structures is this series convergent? Is this the case of relational structures R whose age $\mathcal{A}(R)$ is well-quasi-ordered by embeddability?

Remark 1.4. When $\mathcal{H}_{\varphi_R}$ is a rational fraction of the form above then, for n large enough, $\varphi_R(n)$ is a *quasi-polynomial* of degree k', with $k' \leq k-1$, that is a polynomial $a_{k'}(n)n^{k'} + \cdots + a_0(n)$ whose coefficients $a_{k'}(n), \ldots, a_0(n)$ are periodic functions. Since the profile is non-decreasing, it follows that $a_{k'}(n)$ is eventually constant. Hence the profile has polynomial growth: $\varphi_R(n) \sim an^{k'}$ for some non-negative real a.

With the contribution of P. J. Cameron, this also gives links to some quite venerable fields of mathematics. Indeed, P. J. Cameron [Cam97] associates to the age of R, $\mathcal{A}(R)$, its *age algebra*, a graded commutative algebra $\mathbb{K}.\mathcal{A}(R)$ over a field $\mathbb{K}$ of characteristic zero, and shows that the dimension of the homogeneous component of degree n of $\mathbb{K}.\mathcal{A}(R)$ is $\varphi_R(n)$, hence the generating series above is simply the Hilbert series of $\mathbb{K}.\mathcal{A}(R)$. Other graded algebras than the age algebra enjoy this property. In any case, the association between relational structures and graded algebras via the profile seems to be an interesting topic.

The purpose of this paper is to document this association. We do that for a special case of relational structures that we introduce here for the first time: *those admitting a finite monomorphic decomposition*. Despite the apparent simplicity of

these relational structures, the corresponding age algebras include familiar objects like invariant rings of finite permutation groups. The profile of these relational structures is a quasi-polynomial (cf. Theorem 2.16). This supports the conjecture that the profile of a relational structures with bounded signature or finite kernel is a quasi-polynomial whenever the profile is bounded by some polynomial (for more on the profile, see [Pou02]).

We particularly study the special case of relational structures associated with a permutation groupoid G on a finite set X; as it turns out, their age algebra is a subring of $\mathbb{K}[X]$, the *invariant ring* associated to G (cf. Theorem 4.4). This setting provides a close generalization of invariant rings of permutation groups which includes other famous algebras like quasi-symmetric polynomials and their generalizations. We analyze in details which properties of invariant rings of permutation groups carry over — or not — to permutation groupoids (cf. Propositions 4.18 and 4.9, and Theorems 4.12 and 4.17). To this end, we use in particular techniques from [GS84].

A strong impulse to this research came from the paper by Garsia and Wallach [GW03], which proves that, like invariant rings of permutation groups, the rings of quasi-symmetric polynomials are Cohen-Macaulay. Indeed, a central long term goal is to characterize those permutation groupoids whose invariant ring is Cohen-Macaulay (cf. Problem 4.14), this algebraic property implying that the profile can be written as a quasi-polynomial whose numerator has non-negative coefficients. As a first step in this direction, we analyze several examples, showing in particular that not all invariant rings of permutation groupoids are Cohen-Macaulay.

2. Relational structures admitting a finite monomorphic decomposition

A *monomorphic decomposition* of a relational structure R is a partition $\mathcal{P}$ of E into blocks such that for every integer n, the induced structures on two n-elements subsets A and A' of E are isomorphic whenever the intersections $A \cap B$ and $A' \cap B$ over each block B of $\mathcal{P}$ have the same size.

2.1. **Some examples of relational structures admitting a finite monomorphic decomposition.** .

Example 2.1. Let $R := (\mathbb{Q}, \leq, u_1, \dots, u_{k+1})$ where $\mathbb{Q}$ is the chain of rational numbers, $u_1, \dots, u_{k+1}$ are $k+1$ unary relations which divide Q into $k+1$ intervals. Then $\varphi_R(n) = \binom{n+k}{k}$ and $\mathcal{H}_{\varphi_R} = \frac{1}{(1-Z)^{k+1}}$.

Example 2.2. A graph $G := (V, \mathcal{E})$ being considered as a binary irreflexive and symmetric relation, its profile φ_G is the function which counts, for each integer n, the number $\varphi_G(n)$ of induced subgraphs on n elements subsets of $V(G)$, isomorphic subgraphs counting for one.

Trivially φ_G is constant, equal to 1, if and only if G is a clique or an infinite independent set. A bit less trivial is the fact that φ_G is bounded if and only if G is *almost constant* in the sense of R. Fraïssé [Fra00], that is there is a finite subset F_G of vertices such that two pairs of vertices having the same intersection on F_G are both edges or both non-edges.

Example 2.3. Let G be the direct sum $K_\omega \oplus K_\omega$ of two infinite cliques; then $\varphi_G(n) = \lfloor \frac{n}{2} \rfloor + 1$.

Example 2.4. Let G be the direct sum of $k+1$ many infinite cliques; then $\varphi_G(n) = p_{k+1}(n) \simeq \frac{n^k}{(k+1)!k!}$.

Examples 2.5. Let G be the direct sum $K_{(1,\omega)} \oplus \overline{K}_\omega$ of an infinite wheel and an infinite independent set, or the direct sum $K_\omega \oplus \overline{K}_\omega$ of an infinite clique and an infinite set; then $\varphi_G(n) = n$. Hence $\mathcal{H}_{\varphi_G} = 1 + \frac{Z}{(1-Z)^2}$, that we may write $\frac{1-Z-Z^2}{(1-Z)^2}$, as well as $\frac{1+Z^3}{(1-Z)(1-Z^2)}$.

The first example above has three monomorphic components, one being finite, whereas the second one has two component, both infinite; still, the generating series coincide, with one representation as a rational fraction with a numerator with some negative coefficient, and another with all coefficients non-negative.

A more involved example is the following:

Example 2.6. Let $R := (E, (\rho, U_2, U_3))$, where $E := \mathbb{N}\times\{0,1,2,3\}$, $\rho := \{((n,i),(m,j)) : i = 0, j \in \{1,2\}$ or $i = 1, j = 3\}$; $U_i := \mathbb{N} \times \{i\}$ for $i \in \{0,1,2,3\}$. Then R has four monomorphic components, namely $\mathbb{N}\times\{0\}, \mathbb{N}\times\{1\}, \mathbb{N}\times\{2\}, \mathbb{N}\times\{3\}$. Let S be the induced structure on four elements of the form $(x_i, i), i \in \{0,1,2,3\}$. A crucial property is that S has only two non-trivial local isomorphisms, namely the map sending $(x_0, 0)$ onto $(x_1, 1)$ and its inverse. From this follows that the induced substructures on two n-element subsets E are isomorphic if either they have the same number of elements on each $\mathbb{N}\times\{i\}$ or one subset is included into $\mathbb{N}\times\{0\}$, the other into $\mathbb{N}\times\{1\}$. Hence, the generating series $\mathcal{H}_{\varphi_R}$ is $\frac{1}{(1-Z)^4} - \frac{Z}{1-Z} = \frac{1-Z+3Z^2-3Z^3+Z^4}{(1-Z)^4}$. We may write it $\mathcal{H}_{\varphi_R} = \frac{Q_1}{(1-Z)(1-Z^4)(1-Z^5)(1-Z^5)}$ where $Q_1 := 1+2Z+6Z^2+10Z^3+14Z^4+17Z^5+18Z^6+14Z^7+10Z^8+6Z^9+Z^{10}$, as well as $\mathcal{H}_{\varphi_R} = \frac{Q}{(1-Z)(1-Z^5)^3}$ where $Q_2 := 1+2Z+6Z^2+10Z^3+15Z^4+18Z^5+22Z^6+18Z^7+15Z^8+10Z^9+6Z^{10}+Z^{12}+Z^{16}$.

Here is an example of relational structure R such that $\mathcal{H}_{\varphi_R}$ is a rational fraction, for which there is no way of choosing the numerator with non-negative coefficients.

Example 2.7. Let $R := (E, (\rho_0, \rho_1, \rho_2))$ be defined as follows. First, $E := \mathbb{N} \times \{0,1,2,3\} \setminus \mathbb{N}^* \times \{0,1\}$, that is E is the union of two one-element sets, $E_0 := \{(0,0)\}, E_1 := \{(0,1)\}$, and of two infinite sets, $E_2 := \mathbb{N}\times\{2\}, E_3 := \mathbb{N}\times\{3\}$. Next $\rho_i := E_i \times (E_2 \cup E_3)$ for $i = 0,1$ and $\rho_2 := E_0 \times E_1 \times (E_1 \cup E_2)$. Then R has four monomorphic components, namely $E_0, E_1, E_2 E_3$. The crucial property is that the induced structures on two n-element subsets A, B of E are isomorphic if either A and B includes $E_0 \cup E_1$ and their respective traces on E_2, E_3 have the same size, or A and B include exactly one of the sets E_0 E_1 and not the other, or exclude both. From this $\varphi_R(0) = 1$ and $\varphi_R(n) = n+2$ for $n \geq 1$, hence $\mathcal{H}_{\varphi_R} = \frac{1+Z-Z^2}{(1-Z)^2}$. If $\mathcal{H}_{\varphi_R} = \frac{P}{(1-Z)(1-Z^k)}$ with $k \geq 2$, then

$$P = 1 + 2Z + \sum_{j=2}^{k-2} Z^j - Z^{k+1}$$

hence has a negative coefficient .

Another example with the same property, with two monomorphic component which are both infinite.

Example 2.8. Let $R := (E, \mathcal{H})$, where $E := \mathbb{N}\times\{0,1\}$, $\mathcal{H} := [\mathbb{N}\times\{0\}]^3 \cup [\mathbb{N}\times\{1\}]^3$. Then R has two monomorphic components, namely $\mathbb{N}\times\{0\}$ and $\mathbb{N}\times\{1\}$. Each type

of n-element restriction has a representative made of a $m+k$ element subset of $\mathbb{N}\times\{0\}$ and of a m-element subset of $\mathbb{N}\times\{1\}$ such that $n=2m+k$; these representatives are non-isomorphic, except if $n=2$ (in the later case, all 2 -element restrictions are isomorphic, hence we may eliminate the representative corresponding to $m=1, k=0$). With this observation, a straightforward computation shows that $\varphi_R(0)=\varphi_R(1)=\varphi_R(2)=1$ and $\varphi_R(n)=\lfloor\frac{n}{2}\rfloor+1$ for $n\geq 3$. Hence the generating series $H_{\varphi_R}=\frac{1}{(1-x)(1-x^2)}-x^2=\frac{1-x^2+x^3+x^4-x^5}{(1-x)(1-x^2)}$.

But, then H_{φ_R} cannot be written as a quotient of the form $\frac{P}{(1-x)(1-x^k)}$ where P is a polynomial with non-negative integer coefficients. Indeed, suppose, by contradiction, that H_{φ_R} is of this form. We may suppose k even (otherwise, multiply P and $(1-x)(1-x^k)$ by $(1+x^k)$. Set $k':=\frac{k}{2}$. Multiplying $1-x^2+x^3+x^4-x^5$ and $(1-x)(1-x^2)$ by $1+x^2+\cdots+x^{2(k'-1)}$, we get $P=(1-x^2+x^3+x^4-x^5)(1+x^2+\cdots+x^{2(k'-1)})$. Hence, the term of largest degree has a negative coefficient, a contradiction.

See also Example 4.13 for a last example with three infinite monomorphic components and additional algebraic structure which still has this property.

2.1.1. *Examples coming from group actions.* The *orbital profile* of a permutation group G acting on a set E is the function θ_G which counts for each integer n the number, possibly infinite, of orbits of the n-element subsets of E.

As it is easy to see, orbital profiles are special cases of profiles. Indeed, for every G there is a relational structure such that $\mathrm{Aut}R=\overline{G}$ (the topological closure of G in the symmetric group $\mathfrak{S}(E)$, equipped with the topology induced by the product topology on E^E, E being equipped with the discrete topology). Groups for which the orbital profile takes only finite values are said *oligomorphic*, cf. P. J. Cameron book [Cam90]. These groups are quite common. Indeed, *if G is a group acting on a denumerable set E and R is a relational structure such that* $\mathrm{Aut}R=\overline{G}$ *Then G is oligomorphic if and only if the complete theory of M is $\aleph_0$-categorical (Ryll-Nardzewski, 1959).*

Even in the special case of groups, questions we ask in section 1 have not been solved yet. For an example, let G be a group acting on a denumerable set E; if the orbital profile of G is bounded above by some polynomial, is the generating series of this profile a rational fraction? a rational fraction of the form given in Question 1? with a numerator with non-negative coefficients?

Several examples come from relational structures which decompose into finitely many monomorphic components.

Example 2.9. Let G be is the identity group on a m element set E. Set $R:=(E,U_1,\ldots,U_m)$ where $U_1,\ldots,U_m$ are m-unary relations defining the m elements of E; then $\theta_G(n)=\varphi_R(n)=\binom{m}{n}$.

Example 2.10. Let $G:=\mathrm{Aut}\mathbb{Q}$, where $\mathbb{Q}$ is the chain of rational numbers. Then $\theta_G(n)=\varphi_{\mathbb{Q}}(n)=1$ for all n.

Example 2.11. Let G' be the wreath product $G':=G\wr\mathfrak{S}_{\mathbb{N}}$ of a permutation group G acting on $\{1,\ldots,k\}$ and of $\mathfrak{S}_{\mathbb{N}}$, the symmetric group on $\mathbb{N}$. Looking at G' as a permutation group acting on $E':=\{1,\ldots,k\}\times\mathbb{N}$, then $G'=\mathrm{Aut}R'$ for some relational structure R' on E'; moreover, for all n, $\theta_{G'}(n)=\varphi_{R'}(n)$. Among the possible R' take $R\wr\mathbb{N}:=(E',\equiv,(\overline{\rho}_i)_{i\in I})$ where $\equiv$ is $\{((i,n),(j,m))\in E'^2 : i=j\}$,

$\overline{\rho}_i := \{((x_1, m_1), \ldots, (x_{n_i}, m_{n_i})) : (x_1, \ldots, x_{n_i}) \in \rho_i, (m_1, \ldots, m_{n_i}) \in \mathbb{N}^{n_i}\}$, and $R := (\{1, \ldots, k\}, (\rho_i)_{i \in I})$ is a relational structure having signature $\mu := (n_i)_{i \in I}$ such that $\mathrm{Aut}R = G$. The relational structure $R \wr \mathbb{N}$ decomposes into k monomorphic components, namely the equivalence classes of $\equiv$.

As it turns out, $\mathcal{H}_{\varphi_{R \wr \mathbb{N}}}$ is the Hilbert series $\sum_{n=0}^{\infty} \dim \mathbb{K}[X]_n^G Z^n$ of the invariant ring $\mathbb{K}[X]^G$ of G (that is the subring of the polynomials in the indeterminates $X := (x_1, \ldots, x_k)$ which are invariant under the action of G) (Cameron [Cam90]). As it is well known, this Hilbert series is a rational fraction of the form indicated in Question 1.3, where the coefficients of $P(Z)$ are non-negative.

Problem 2.12. Find an example of a permutation group G' acting on a set E with no finite orbit, such that the orbital profile of G' has polynomial growth, but the generating series is not the Hilbert series of the invariant ring $\mathbb{K}[X]^G$ of a permutation group G acting on a finite set X.

2.1.2. *Quasi-symmetric polynomials and the like.*

Example 2.13. Let $X_k := (x_1, \ldots, x_k)$ be k indeterminates and $n_1, \ldots, n_l$ be a sequence of positive integers, $l \leq k$. The polynomial

$$\sum_{1 \leq i_1 < \cdots < i_l \leq k} x_{i_1}^{n_1} \ldots x_{i_l}^{n_l}$$

is a *quasi-symmetric monomial* of degree $n := n_1 + \cdots + n_l$. The vector space spanned by the quasi-symmetric monomials forms the space $\mathrm{QSym}(X_k)$ of *quasi-symmetric polynomials* as introduced by I. Gessel. As in the example above, the Hilbert series of $\mathrm{QSym}(X_k)$ is defined as

$$\mathcal{H}_{\mathrm{QSym}(X_k)} := \sum_{n=0}^{\infty} \dim \mathrm{QSym}(X_k)_n Z^n \ .$$

As shown by F. Bergeron and C. Reutenauer (cf. [GW03]), this is a rational fraction of the form $\frac{P_k}{(1-Z)*(1-Z^2)*\ldots(1-Z^k)}$ where the coefficients P_k are non negative. Let R be the poset product of a k-element chain by a denumerable antichain. More formally, $R := (E, \rho)$ where $E := \{1, \ldots, k\} \times \mathbb{N}$ and $\rho := \{((i, n), (j, m)) \in E$ such that $i \leq j\}$. Each isomorphic type of an n-element restriction may be identified to a quasi-symmetric polynomial, hence the generating series associated to the profile of R is the Hilbert series defined above.

Example 2.14. A relational structure $R := (E, (\rho_i)_{i \in I})$ is *categorical for its age* if every R' having the same age as R is isomorphic to R. It was proved in [HM88] that for relational structure with finite signature (I finite) this happens just in case E is countable and can be divided into finitely many blocks such that every permutation of E which preserves each block is an automorphism of R.

2.2. **Results and problems about relational structures admitting a finite monomorphic decomposition.** The following result motivates the introduction of the notion under review.

Theorem 2.15. *The profile of a relational structure R is bounded by some integer if and only if R has a monomorphic decomposition into finitely many blocks, at most one being infinite.*

Relational structures satisfying the second condition of the above sentence are the so-called *almost-monomorphic relational structures* of R. Fraïssé. Theorem 2.15 above was proved in [FP71] for finite signature and in [Pou81]) for arbitrary signature by means of Ramsey theorem and compactness theorem of first order logic. From Theorem 1.1 and Theorem 2.15, it follows that a relational structure R has a monomorphic decomposition into finitely many blocks, at most one being infinite if and only if

$$\mathcal{H}_{\varphi_R} = \frac{1 + b_1 Z + \cdots + b_l Z^l}{1 - Z} ,$$

where $b_1, \ldots, b_l$ are non negative integers.

It is trivial that, if an infinite relational structure R has a monomorphic decomposition into finitely many blocks, whereof k are infinite, then the profile is bounded by some polynomial, whose degree itself is bounded by $k - 1$.

Theorem 2.16. *Let R be an infinite relational structure R with a monomorphic decomposition into finitely many blocks $(E_i, i \in X)$, k of which being infinite. Then, the generating series $\mathcal{H}_{\varphi_R}$ is a rational fraction of the form:*

$$\frac{P(Z)}{(1 - Z)(1 - Z^2) \cdots (1 - Z^k)} .$$

In particular, remark 1.4 applies.

To each subset A of size d of E, we associate the monomial

$$x^{d(A)} := \prod_{i \in X} x_i^{d_i(A)} ,$$

where $d_i(A) = |A \cap E_i|$ for all i in X. Obviously, A is isomorphic to B whenever $x^{d(A)} = x^{d(B)}$. The shape of a monomial $x^d = \prod x_i^{d_i}$ is the partition obtained by sorting decreasingly $(d_i, i \in X)$. We define a total order on monomials by comparing their shape w.r.t. the degree reverse lexicographic order, and breaking ties by the usual lexicographic order on monomials w.r.t. some arbitrary fixed order on X. To each orbit of sets, we associate the unique maximal monomial $\mathrm{lm}(A)$, where A ranges through the orbit; we call this monomial *leading monomial*. To prove the theorem, we essentially endow the set of leading monomials with an ideal structure in some appropriate polynomial ring. This is reminiscent of the chain-product technique as defined in Subsection 4.1.1. The key property of leading monomials is this:

Lemma 2.17. *Let m be a leading monomial, and $S \subset X$ a layer of m. Then, either $d_i = |E_i|$ for some i in S, or mx_S is again a leading monomial.*

The proof of this result relies on Proposition 2.19 below for which we introduce the following definition. Let R be a relational structure on E; a subset B of E is a *monomorphic part* of R if for every integer n and every pair A, A' of n-element subsets of E the induced structures on A and A' are isomorphic whenever $A \setminus B = A' \setminus B$. The following lemma, given without proof, rassembles the main properties of monomorphic parts.

Lemma 2.18. *(i) The emptyset and the one element subsets of E are monomorphic parts of R;*

(ii) If B is a monomorphic part of R then every subset of B too;

(iii) *Let B and B' be two monomorphic parts of R; if B and B' intersect, then $B \cup B'$ is a monomorphic part of R;*

(iv) *Let $\mathcal{B}$ be a family of monomorphic parts of R; if $\mathcal{B}$ is up-directed (that is the union of two members of $\mathcal{B}$ is contained into a third one), then their union $B := \bigcup \mathcal{B}$ is a monomorphic part of R.*

Let $x \in E$, let $R(x)$ be the set-union of all the monomorphic parts of R containing x. By (i) of Lemma 2.18 this set contains x and by (iii) and (iv) this is a monomorphic part, thus the largest monomorphic part of R containing x.

Proposition 2.19. *The largest monomorphic parts form a monomorphic decomposition of R off which every monomorphic decomposition of R is a refinement.*

Proof of Lemma 2.17. Let $e := |X|$, $\overline{d} := (d_{i_1}, \dots, d_{i_e})$ be the shape of m sorted decreasingly and $s := |S|$. Suppose that $d_i < |E_i|$ for every i in S. Let A, B , B' be subsets of E such that $x^{d(A)} = m$, $x^{d(B)} = mx_S$, and $m' := x^{d(B')}$ is the leading monomial in the orbit of B and let R_A, R_B, $R_{B'}$ be the corresponding induced structures.

Clearly, the shape of mx_S is $\overline{d_1} := (d_{i_1} + 1, \dots, d_{i_s} + 1, d_{i_{s+1}}, \dots, d_{i_e})$. Let $\overline{d'} := (d'_{i'_1}, \dots, d'_{i'_s}, d'_{i'_{s+1}}, \dots, d'_{i'_e})$ be the shape of B'. Our first goal is to prove that these two shapes are the same.

Claim 1 $d_{i_p} = d'_{i'_p}$ *for all $p > s$.*

Proof of Claim 1 Suppose this does not hold. Let p be the largest such that $d_{i_p} \neq d'_{i'_p}$. Since, by definition, we have $\overline{d'} \geq \overline{d_1}$ it follows that $d'_{i'_p} < d_{i_p}$; thus, $\overline{d'} > \overline{d}$. However, since $R_{B'}$ contains a copy of R_A we have $\overline{d'} \leq \overline{d}$, a contradiction. □

Set $U := \bigcup\{E_i \cap B : i \notin S\}$, $S' := \{i'_1, \dots, i'_s\}$, $U' := \bigcup\{E_i \cap B' : i \notin S'\}$. Let φ be an isomorphism from R_B onto $R_{B'}$.

Claim 2 *U is the set of $x \in B$ such that the induced structure $R_{B\setminus\{x\}}$ on $B\setminus\{x\}$ contains no copy of R_A. Moreover φ transforms U into U'.*

Proof of Claim 2 From the definition of U, $R_{B\setminus\{x\}}$ contains a copy of R_A for every element $x \in B \setminus U$. Conversely, let $x \in U$ and let $\overline{d''}$ be the shape of $B \setminus \{x\}$. Clearly, for the largest p such that $d''_{i''_p} \neq d_{i_p}$ we have $p > s$. Hence $\overline{d}'' > \overline{d}$, thus $R_{B\setminus\{x\}}$ cannot contains a copy of R_A. This proves the first part of Claim 2.

Since, from Claim 1, $d_{i_p} = d'_{i'_p}$ for all $p \geq s$, the same argument show that if $x' \in U'$ then $R_{B'\setminus\{x'\}}$ cannot contains a copy of R_A. Since from Claim 1, U and U' have the same size we get that U' is the set of $x' \in B'$ such that $R_{B'\setminus\{x'\}}$, contains no copy of R_A. The second part of Claim 2 follows immediately.

Claim 3 *Let $i \notin S$ and $j \in S$ then every monomorphic part containing $E_i \cap B$ is disjoint from $E_j \cap B$.*

Proof of Claim 3 According to Claim 2 a monomorphic part containing $E_i \cap B$ must be disjoint from U.

Claim 4 For each $i \in S$, $E_i \cap B$ is a largest monomorphic part of R_B.

Proof of Claim 4 Suppose not. Then this largest monomorphic part, say C, contains some other $E_j \cap B$. From Claim 3, $j \in S$. It follows that all induced substructures on $C \setminus \{x, y\}$, where $\{x, y\}$ is a pair of distinct elements of C, are isomorphic. Suppose $d_i \geq d_j$. Since the shape of A is maximal then for $x, y \in E_j$

the induced structure does not contain a copy of R_A. But if $x \in E_i$ and $y \in E_j$ then trivially the induced structure contains a copy of R_A. A contradiction.

Claim 5 *φ transforms $(E_i \cap B, i \in S)$ into $(E_i \cap B', i \in S')$*

Proof of Claim 5. The $E_i \cap B$'s for $i \in S$ are the largest monomorphic parts of $R_{B \setminus A}$. Via φ there are transformed into the s largest monomorphic parts of $R_{B' \setminus U'}$. Since $(E_i \cap B', i \in S')$ is a decomposition of $R_{B' \setminus U'}$ into s monomorphic parts, this decomposition coincides with this decomposition into largest parts.

From Claim 1 and Claim 5, we have $\overline{d'} = \overline{d_1}$. Suppose that $m' > mx_S$. Let T be a transversal of the $E_i \cap B$'s for $i \in S$. Then, from Claim 5, $T' := \varphi(T)$ is a transversal of the $E_{i'} \cap B'$'s for $i' \in S'$. Let m_T, resp. $m_{T'}$, be the monomial associated with $B \setminus T$, resp. $B' \setminus T'$. We have $m'_{T'} > m_T$. Since $m_T = m$ and $B' \setminus T'$ is in the orbit of B, we get a contradiction.

Proof of theorem 2.16. Fix a chain $C = (\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_r \subset X)$ of non empty subsets of X. Let lm_C be the set of leading monomials with chain support C. The plan is essentially to realize lm_C as the linear basis of some ideal of a polynomial ring, so that the generating series of lm_C is realized as an Hilbert series.

Consider the polynomial ring $\mathbb{K}[S_1, \ldots, S_l]$, with its natural embedding in $\mathbb{K}[X]$ by $S_j \mapsto \prod_{i \in S_j} x_i$. Let I be the subspace spanned by the monomials $m = S_1^{r_1} \ldots S_l^{r_l}$ such that $d_i(m) > |E_i|$ for some i; it is obviously a monomial ideal. When all monomorphic components are infinite, I is the trivial ideal $\{0\}$. Consider the subspace $\mathbb{K}.\mathrm{lm}_C$ of $\mathbb{K}[S_1, \ldots, S_l]$ spanned by the monomials in lm_C. Lemma 2.17 exactly states that $J = \mathbb{K}.\mathrm{lm}_C \oplus I$ is in fact a monomial ideal of $\mathbb{K}[S_1, \ldots, S_l]$. Both I and J have finite free resolution as modules over $\mathbb{K}[S_1, \ldots, S_l]$, so that their Hilbert series are rational fractions of the form:

$$\frac{P}{(1 - Z^{|S_1|}) \cdots (1 - Z^{|S_l|})} .$$

Hence, the same hold for $\mathcal{H}_{\mathbb{K}.\mathrm{lm}_C} = \mathcal{H}_J - \mathcal{H}_I$. Furthermore, whenever S_j contains i with $|E_i| < \infty$, the denominator $(1 - Z^{|S_l|})$ can be canceled out in $\mathcal{H}_{\mathbb{K}.\mathrm{lm}_C}$. The remaining denominator divides $(1 - Z) \cdots (1 - Z^k)$.

By summing up those Hilbert series $\mathcal{H}_{\mathbb{K}.\mathrm{lm}_C}$ over all chains C of subsets of X, we get the generating series of all the leading monomials, that is the profile of R. Hence, this profile is a rational fraction of the form:

$$\mathcal{H}_{\mathbb{K}.\mathrm{lm}_C} = \frac{P}{(1 - Z) \cdots (1 - Z^k)} .$$

□

Remark 2.20. As Examples 2.8 and 4.13 illustrates, it is not true that if all blocks of a monomorphic decomposition of R are infinite, then the numerator P in the above fraction can be choosen with non-negative coefficients.

We do not know for which relational structures having a finite monomorphic decomposition the numerator P can be choosen with non-negative coefficients. A possible approach is to look for some sensible Cohen-Macaulay graded algebra whose Hilbert series is $\mathcal{H}_{\varphi_R}$ (by proposition 4 of [BM04] such a Cohen-Macaulay algebra always exists as soon as P has non-negative coefficients). This is one of our motivations for the upcoming study of the *age algebras*.

3. The age algebra of a relational structure

3.1. **The set-algebra.** Let E be a set and let $[E]^{<\omega}$ be the set of finite subsets of E (including the empty set). Let $\mathbb{K}$ be a field, and $\mathbb{K}^{[E]^{<\omega}}$ be the set of maps $f : [E]^{<\omega} \to \mathbb{K}$. Endowed with the usual addition and scalar multiplication of maps, $\mathbb{K}^{[E]^{<\omega}}$ is a $\mathbb{K}$-vector space. Let $f, g \in \mathbb{K}^{[E]^{<\omega}}$; according to Cameron, we set:

$$fg(P) := \sum_{M \in [P]^{<\omega}} f(P)g(P \setminus M) \ ,$$

for all $P \in [E]^{<\omega}$ [Cam97]. With this operation added, $\mathbb{K}^{[E]^{<\omega}}$ becomes a ring. This ring is commutative and has a unit: denoted by 1, this is the map taking the value 1 on the empty set and the value 0 everywhere else.
Let $\equiv$ be an equivalence relation on $[E]^{<\omega}$. A map $f : [E]^{<\omega} \to \mathbb{K}$ is $\equiv$-*invariant* or, briefly, *invariant* if f is constant on each equivalence class. Invariant maps form a subspace of the vector space $\mathbb{K}^{[E]^{<\omega}}$. We give a condition below which insures that they form a subalgebra too.

Lemma 3.1. *Let $\equiv$ be an equivalence relation on $[E]^{<\omega}$ and $D, D' \in [E]^{<\omega}$. Then the following properties are equivalent:*

(i) There exists some bijective map $f : D \hookrightarrow\!\!\!\!\rightarrow D'$ such that $D \setminus \{x\} \equiv D' \setminus \{f(x)\}$ for every $x \in D$;
(ii) 1) $|D| = |D'| = d$ for some d;
2) $|\{X \in [D]^{d-1} : X \equiv B\}| = |\{X \in [D']^{d-1} : X \equiv B\}|$ for every $B \subseteq E$.

An equivalence relation on $[E]^{<\omega}$ is *hereditary* if every pair D, D' of equivalent elements satisfies one of the two equivalent conditions of Lemma 3.1.

Remark 3.2. Hereditary equivalences are introduced in [PR86] with Condition (ii) 2) of Lemma 3.1 replaced by the condition:

$$|\{X \subseteq D : X \equiv B\}| = |\{X \subseteq D' : X \equiv B\}| \ \text{ for every } B \subseteq E \ .$$

It follows from the next Lemma that this condition is not stronger.

Let $\equiv$ be an equivalence relation on $[E]^{<\omega}$. We denote by $[E]^{<\omega}_{/\equiv}$ the set of equivalence classes. Let $a, b, c \in [E]^{<\omega}_{/\equiv}$ and $D \in [E]^{<\omega}$. Set

$$\chi_{a,b,c}(D) := |\{(A, B) \in a \times b : A \cup B = C, C \subseteq D, C \in c\}| \ .$$

If all subsets of E belonging to some equivalence class a have the same size, we denote by $|a|$ this common size.

Lemma 3.3. *If $\equiv$ is an hereditary equivalence relation on $[E]^{<\omega}$ then*

$$\chi_{a,b,c}(D) := \chi_{a,b,c}(D') \text{ whenever } \ D \equiv D'. \tag{1}$$

Proposition 3.4. *Let $\equiv$ be an hereditary equivalence relation on $[E]^{<\omega}$. Then the product of two invariant maps is invariant.*

3.2. **The age algebra.** Let R be a relational structure with domain E. Set $F \equiv F'$ for $F, F' \in [E]^{<\omega}$ if the restrictions $R\restriction_F$ and $R\restriction_{F'}$ are isomorphic. The resulting equivalence on $[E]^{<\omega}$ is hereditary, hence the set of invariant maps $f : [E]^{<\omega} \to \mathbb{K}$ form a subalgebra of $\mathbb{K}^{[E]^{<\omega}}$. Let $\mathbb{K}.\mathcal{A}(R)$ be the subset made of the invariant maps which are everywhere zero except on a finite number of equivalence classes. Then $\mathbb{K}.\mathcal{A}(R)$ forms an algebra, the *age algebra* of Cameron.

4. Invariant rings of permutation groupoids

Let $\mathbb{K}$ be a field of characteristic 0. In this section, we study in more details a specific class of age algebras which can be realized as graded subrings of polynomial rings $\mathbb{K}[X]$ that we call *invariant rings of permutation groupoids.* This class extends the class of invariant rings of permutation groups (Example 2.11), and contains other interesting examples like the rings of quasi-symmetric polynomials (Example 2.13). Our long-term motivations are twofold. On one hand, relate, in this simpler yet rich setting, the properties of the profile to algebraic properties of the invariant ring. In particular, find conditions under which the invariant ring is Cohen-Macaulay. On the other hand, generalize the theory, algorithms, and techniques of invariant rings of permutation groups to a larger class of subrings of $\mathbb{K}[X]$. In particular, find new properties of the ring of quasi-symmetric polynomials, one specific goal being to find a simpler proof that this ring is Cohen-Macaulay.

4.1. **Permutation groupoids.** Let X be a finite set. A *local bijection* of X is a bijective function $f : \mathrm{dom} f \hookrightarrow\!\!\!\!\rightarrow \mathrm{im} f$ whose domain $\mathrm{dom} f$ and image $\mathrm{im} f$ are subsets of X. The *rank* of f is the size of its domain, so that f is a permutation of X if it is of maximal rank $|X|$. The inverse f^{-1} of a local bijection f, its restriction $f_{\restriction X'} : X' \hookrightarrow\!\!\!\!\rightarrow f(X')$ to a subset X' of $\mathrm{dom} f$, and the composition $f \circ g$ of two local bijections f and g such that $\mathrm{im} g = \mathrm{dom} f$ are defined in the natural way. A set G of local bijections of X is called a *permutation groupoid* if it contains the identity and is stable by restriction, inverse, and composition. It can be seen as a category: the objects are the subsets of X and the morphisms $G(A, B)$ from A to B are the local bijections $f : A \hookrightarrow\!\!\!\!\rightarrow B$ in G. Those morphisms are by definition isomorphisms, and G satisfies the usual *groupoid* axioms.

The *underlying permutation group* is the subset $G(X, X)$ of all permutations in G; those are exactly the invertible elements w.r.t. the composition product.

Examples 4.1. The set $\downarrow\mathfrak{S}(X)$ of all local bijections of X is a permutation groupoid.

The closure $\downarrow G$ of a permutation group G by restriction is a permutation groupoid. In the following, we say that $\downarrow G$ *comes from the permutation group* G.

Let $X := \{1, \dots, n\}$. The set G of strictly increasing local bijections of X forms a permutation groupoid. Obviously G does not come from a permutation group since its underlying permutation group is reduced to the identity.

Let R be a relational structure on X. The local isomorphisms of R form a permutation groupoid. Its underlying permutation group is the automorphism group of R. Typically, the previous example is obtained by taking as relational structure R the chain $1 < 2 < \cdots < n$. Also, $\downarrow\mathfrak{S}(X)$ is obtained by taking the trivial relational structure on X. In fact, *any permutation groupoid G can be obtained from a suitable relational structure R_G on X* (recall that X is finite!).

4.1.1. *The invariant ring of a permutation groupoid.* Let G be a permutation groupoid acting on a finite set X, and $\mathbb{K}[X]$ be the polynomial ring whose variables x_i are indexed by the elements i of X. Given a local function f of G, and a monomial $x^d := \prod_{i \in X} x_i^{d_i}$ whose support $\mathrm{support}(x^d)$ is contained in $\mathrm{dom} f$, we set

$$f.x^d := \prod_{i \in X, d_i > 0} x_{f(i)}^{d_i} \ ,$$

generalizing the usual action of a permutation on a monomial. This partial action of G on monomials does not extend to a global action of G. Still, notions like

G-isomorphic monomials and *G-orbits* are well defined. The *orbit sum* $o(x^d)$ of a monomial x^d is the sum of all the monomials in its orbit.

Our object of study is the *invariant ring* $\mathbb{K}[X]^G$ of G, which is defined as the linear subspace of $\mathbb{K}[X]$ spanned by the orbitsums of all monomials.

Examples 4.2. Let G be a permutation group. Then, $\mathbb{K}[X]^{\downarrow G}$ is the usual invariant ring of G.

Let G be the permutation groupoid of the strictly increasing local bijections of $\{1,\dots,n\}$. Then $\mathbb{K}[X]^G$ is the ring $\mathrm{QSym}(X_n)$ of quasi-symmetric polynomials on the ordered alphabet $X_n := (x_1,\dots,x_n)$.

Taking the same groupoid G as in the previous example, and letting it act naturally on respectively pairs, couples, k-subsets, or k-tuples of elements of $\{1,\dots,n\}$, yields respectively the (un)oriented (hyper)graph quasi-symmetric polynomials of [NTT04].

Remarks 4.3. The orbitsums form a linear basis of $\mathbb{K}[X]^G$.

It is not obvious from the definition that $\mathbb{K}[X]^G$ is indeed a graded algebra. In the following we prove this by making G into a monoid and the action of G on polynomials into a multiplicative linear representation of G. An other way is to encode G by some relational structure R_G on X and, as in Example 2.11, to define a relational structure $R_G \wr \mathbb{N}$ on $E := X \times \mathbb{N}$ with monomorphic components $E_i := \{i\} \times \mathbb{N}$ for $i \in X$. Let $\phi : \mathbb{K}[X] \hookrightarrow \mathbb{K}^{[X\times\mathbb{N}]^{<\omega}}$ defined by setting $\phi(x^d) := d!\chi_{O_{\mathfrak{G}}(x^d)}$, where $d! := \prod_{i\in X} d_i!$, and $\chi_{O_{\mathfrak{G}}(x^d)}$ is the characteristic function of $O_{\mathfrak{G}}(x^d) := \{A \subset X \times \mathbb{N},\, |A \cap E_i| = d_i, \forall i \in X\}$. Once $\mathbb{K}^{[X\times\mathbb{N}]^{<\omega}}$ is equipped with its set-algebra stucture, ϕ is a morphism of algebras. Applying Proposition 3.4, we get:

Theorem 4.4. *The invariant ring $\mathbb{K}[X]^G$ is isomorphic via ϕ to the age algebra $\mathbb{K}.\mathcal{A}(R_G \wr \mathbb{N})$. In particular, the generating series of the orbits, the generating series of the profile of $R_G \wr \mathbb{N}$, and the Hilbert series of $\mathbb{K}[X]^G$ coincide.*

4.1.2. *Restrictions of permutation groupoids.* The *restriction* $G_{\restriction X'}$ of a permutation groupoid G to a subset X' is the set of all local functions f in G such that $\mathrm{dom} f \subset X'$ and $\mathrm{im} f \subset X'$, which is again a permutation groupoid. Furthermore, the orbits of monomials in $\mathbb{K}[X']$ are unchanged by this restriction. In particular, the invariant ring of $G_{\restriction X'}$ is simply the quotient of the invariant ring of G obtained by killing all the variables x_i with $i \notin X'$. This simple fact is one of the points of considering permutation groupoids instead of just permutation groups (for which the restriction to a subset is not clearly defined). This may indeed give opportunities for induction techniques on the size of the underlying set.

Proposition 4.5. *Any permutation groupoid comes from the restriction of a permutation group of some superset. However, this superset may need to be infinite.*

Examples 4.6. (a) The permutation groupoid on $\{1,2,3\}$ generated by the rank 1 local bijection $1 \mapsto 2$ is the restriction of the permutation group on $\{1,2,3,4\}$ generated by the permutation $(1,2)(3,4)$.

(b) The local automorphism permutation groupoid of the chain $a < b$ is the restriction of the cyclic group C_3 on $\{a,b,c\}$.

(c) Consider a relational structure R such that there exists three elements a,b,c and a binary relation $<$ which restricts on $\{a,b,c\}$ to the chain $a < b < c$. Typically, R is a chain of length at least 3 (giving $\mathrm{QSym}(X)$ as invariants) or a poset of height at least 3. Then, there exists no relational structure $\overline{R}$ on a finite superset where all local isomorphisms extend to global isomorphisms.

4.1.3. *The monoid of a permutation groupoid.* The goal is now to turn G into a monoid, and to make the partial actions of G into a linear representations of this monoid. The composition of two local functions f and g can be extended when $\mathrm{im}f \neq \mathrm{im}g$ by setting it to the local function with the largest domain on which $f(g(x))$ is well-defined:

$$f \circ g : \begin{cases} g^{-1}(\mathrm{im}g \cap \mathrm{dom}f) & \hookrightarrow\!\!\!\!\rightarrow f(\mathrm{im}g \cap \mathrm{dom}g) \\ x \mapsto f(g(x)) \end{cases} .$$

With this composition product, G turns into a monoid whose unit is the identity of X. Now we can extend the partially defined action of local bijections on monomials into a linear action on polynomials by setting:

$$f.x^d := \begin{cases} \prod_{i \in X, d_i > 0} x_{f(i)}^{d_i} & \text{if } \mathrm{support}(x^d) \subset \mathrm{dom}f, \\ 0 & \text{otherwise.} \end{cases}$$

We leave it as exercise to check that this defines a linear representation of the monoid G, which is *multiplicative*: for any f in G and P and Q in $\mathbb{K}[X]$,

$$f.(PQ) = (f.P)(f.Q) .$$

Corollary 4.7. *The invariant ring $\mathbb{K}[X]^G$ is, as its name suggests, indeed a ring.*

Proof. Consider a product of two orbitsums $o(m_1)o(m_2)$, and take two isomorphic monomials m and $f.m$, $f \in G$. Whenever m occurs as a product $m = m_1' m_2'$, $m_1' \in G.m_1$, $m_2' \in G.m_2$, the monomial $f.m$ occurs simultaneously as the product $f.m = f.(m_1' m_2') = f.m_1' f.m_2'$, and reciprocally. Hence m and m' occur with the same coefficient in $o(m_1)o(m_2)$. □

Note that the monoid algebra of G is isomorphic to its groupoid algebra $\mathbb{K}.G$ which is semi-simple. This linear representation of G extends into a linear representation of $\mathbb{K}.G$.

4.1.4. *Groupoid and monoid algebra of a permutation groupoid.* Let G be a permutation groupoid, and $\mathbb{K}$ a field (of characteristic zero; typically $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). By definition, its *groupoid algebra* $\mathbb{K}.G$ is the $\mathbb{K}$-vector space whose basis $\{\mathrm{gr}f, f \in G\}$ is indexed by the elements f of G, and whose product is given by:

$$\mathrm{gr}f\ \mathrm{gr}g = \begin{cases} f \circ g & \text{if } \mathrm{im}g = \mathrm{dom}f, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\{\mathrm{gr}f, f \in G\}$ the *graded basis* of $\mathbb{K}.G$. Similarly the *monoid algebra* of G is defined as the $\mathbb{K}$-vector with basis $\{f, f \in G\}$ equipped with the extended composition product $\circ$.

Remarks 4.8. As the notation suggests, the groupoid and the monoid algebra of G are isomorphic.

The groupoid algebra $\mathbb{K}.G$ is semi-simple, and decomposes as a direct sum of non-unitary algebras:

$$\mathbb{K}.G = \sum_{k=0}^{n} \mathbb{K}.G_k ,$$

where $G_k := \{f \in G, \mathrm{rank}f = k\}$.

Proof. The isomorphism from the monoid algebra to the groupoid algebra is given by:

$$f \mapsto \sum_{A \subset \mathrm{dom} f} \mathrm{gr} f_{\upharpoonright A} \ .$$

The inverse isomorphism is obtained by Möbius inversion:

$$\mathrm{gr} f \mapsto \sum_{A \subset \mathrm{dom} f} (-1)^{|\mathrm{dom} f| - |A|} f_{\upharpoonright A} \ .$$

Checking the compatibility with the product rule is straightforward.

The semi-simplicity is a general property of groupoid algebras, which is easily checked using Dickson's Lemma. □

The linear representations of the monoid G on polynomials extend directly to linear representations of its algebra $\mathbb{K}.G$. In particular, this defines the actions of the graded basis. Its characteristic is that $\mathrm{gr} f$ kills all monomials whose support is not exactly $\mathrm{dom} f$, whereas f kills only those monomials whose support is not contained in $\mathrm{dom} f$.

Important note: the action of $\mathrm{gr} f$ is *not* multiplicative on polynomials! Take for example $f := \mathrm{id}_{\{1,2\}}$, $P := x_1$ and $Q := x_2$. This is in fact the main reason for considering the monoid algebra and not just only the groupoid algebra.

4.2. **Invariants of permutation groupoids.** In this section, we review which properties of invariants of permutation groups extend to permutation groupoids.

4.2.1. *The Reynolds operator.* The first essential feature of invariant rings is the so-called *Reynolds operator*, which is a projector on the invariant ring. The following proposition states that this operator still exists for invariants of permutation groupoids, albeit missing the important property of being a $\mathbb{K}[X]^G$-module morphism. In particular, although $\mathbb{K}[X]^G$ still contains the ring of symmetric polynomials $\mathrm{Sym}(X)$, R is not anymore a $\mathrm{Sym}(X)$-module morphism.

Proposition 4.9. *There exists an idempotent R in the groupoid algebra $\mathbb{K}.G$ which projects $\mathbb{K}[X]$ on the invariant ring $\mathbb{K}[X]^G$:*

$$R := \sum_{A \subset X} \frac{1}{|g \in G, \mathrm{dom} g = A|} \sum_{g \in G, \mathrm{dom} g = A} \mathrm{gr} g \ .$$

Furthermore, the four following conditions are equivalent: R is a $\mathrm{Sym}(X)$-module morphism, R is a $\mathbb{K}[X]^G$-module morphism, $\ker R$ is a Sym-module, and G comes from a permutation group.

4.2.2. *The chain product.* We now define another product $\star$ on the invariant ring $\mathbb{K}[X]^G$, called the *chain product*, which preserves a finer grading. In fact, $(\mathbb{K}[X]^G, \star)$ is a simple realization of the Stanley-Reisner ring of a suitable poset. Such rings have been studied intensively, in particular by Garsia and Stanton [GS84] to construct $\mathrm{Sym}(X)$-module generators for the invariant rings of certain permutation groups, and prove the degree bound for permutation groups $\beta(G) \leq \binom{|X|}{2}$ (recall that the *degree bound* $\beta(A)$ of a finitely generated graded algebra A is the smallest integer such that A is generated by its elements of degree at most $\beta(A)$). This tool is characteristic free: all statements below actually hold over any ground ring.

Given a subset S of X, set $x_S := \prod_{i\in S} x_i$. By square-free decomposition, any monomial x^d can be identified uniquely with a *multichain* $S_1 \subset \cdots \subset S_k$ of nested subsets of X, so that:

$$x^d = x_{S_1} \dots x_{S_k} \ .$$

We call each S_k a *layer* of x. The *fine degree* of the monomial x^d is the integer vector $(r_1, \dots, r_n)$ where each r_i counts the (possibly null) number of repetitions of the layer of size i in x^d. The fine degree defines a filtration on $\mathbb{K}[X]$. The *chain product* $\star$ of two monomials $x^d = x_{S_1} \dots x_{S_k}$ and $x^{d'} = x_{S'_1} \dots x_{S'_k}$ is defined by:

$$x^d \star x^{d'} := \begin{cases} x^d x^{d'} & \text{if } \{S_1, \dots, S_k, S'_1, \dots, S'_k\} \text{ is again a multichain of subsets,} \\ 0 & \text{otherwise.} \end{cases}$$

For example, $x_1 \star x_1 = x_1^2$, $x_1 \star x_2 = 0$, $x_1x_3^2 \star x_1x_2x_3^2 = x_1^2x_2x_3^4$, and $x_1x_3^2 \star x_1x_2 = 0$.

The chain product endows $(\mathbb{K}[X], \star)$ with a second algebra structure (in fact $(\mathbb{K}[X], \star)$ is isomorphic to the quotient $\mathbb{K}[x_S, S \subset X]/\{x_S x_{S'} = 0, S \not\subset S'$ and $S' \not\subset S\})$. It is also finely graded (fine degrees being added term-by-term). In fact, $(\mathbb{K}[X], \star)$ is exactly the associated graded algebra of $\mathbb{K}[X]$ w.r.t. the fine degree filtration. Beware that $(\mathbb{K}[X], \star)$ is not an integral domain.

The elementary symmetric functions

$$e_d := \sum_{S \subset X, |S| = d} x_S$$

are still algebraically independent and generate $(\mathrm{Sym}(X)_n, \star)$. Note that this does not hold for, say, the symmetric powersums. The following simple fact turns out to be an essential key:

Remark 4.10. Consider the chain product of a monomial $x_{S_1} \dots x_{S_k}$ by the elementary symmetric function e_d. It is the sum of all monomials $x_{S_1} \dots x_S \dots x_{S_k}$, where S is of size k, and fits in the chain $S_1 \subset \cdots \subset S \subset \cdots \subset S_k$. In particular, if $x_{S_1} \dots x_{S_k}$ readily contains a layer S of size k, then $x_{S_1} \dots x_{S_k} \star e_k$ is the unique monomial obtained by replicating this layer.

More generally, $(\mathbb{K}[X]^G, \star)$ is a subring of $(\mathbb{K}[X], \star)$. In particular, $(\mathbb{K}[X]^G, \star)$ is a $\mathrm{Sym}(X)$-module. Furthermore, we may transfer the following algebraic properties from $(\mathbb{K}[X], \star)$ to $\mathbb{K}[X]^G$, as in the case of permutation groups [GS84].

Proposition 4.11.
(a) *A family F of finely homogeneous invariants of positive degree which generates $(\mathbb{K}[X]^G, \star)$, also generates $\mathbb{K}[X]^G$;*
(b) *$\beta(\mathbb{K}[X]^G, \star) \geq \beta(\mathbb{K}[X]^G)$;*
(c) *A family F of finely homogeneous invariants which generates $(\mathbb{K}[X]^G, \star)$ as a $\mathrm{Sym}(X)$-module also generates $\mathbb{K}[X]^G$ as a $\mathrm{Sym}(X)$-module;*
(d) *If $(\mathbb{K}[X]^G, \star)$ is a free $\mathrm{Sym}(X)$-module, then so is $\mathbb{K}[X]^G$.*

Proof. This is a standard fact about filtrations and associated graded connected algebras. The key of the proof is that, if p and q are finely homogeneous, the maximal finely homogeneous component of pq is exactly $p \star q$. (a) and (c) follow by induction over the fine grading. Then, (b) follows straightaway from (a), and (d) from (c) by a simple Hilbert series argument. □

The converse of (a) and (b) do not hold. In fact, with most permutation groups, the degree bound $\beta(\mathbb{K}[X]^G, \star)$ is much larger than $\beta(\mathbb{K}[X]^G)$. We conjecture that the converse of (c) and (d) hold. However (d) does not hold anymore

in a slightly larger setting which includes the r-quasi-symmetric polynomials of F. Hivert [Hiv04], a counter example being $\mathrm{QSym}^2(X_3)$ (there is an obstruction in the fine Hilbert series).

Theorem 4.12. *The invariant ring $\mathbb{K}[X]^G$ is a finitely generated algebra and $\mathrm{Sym}(X)$-module, in degree at most $\frac{|X|(|X|+1)}{2}$. This degree bound is tight.*

Note that, as usual, when G does not act transitively on the variables, the degree bound can be greatly improved by considering the elementary symmetric polynomials on each transitive component instead.

Proof. The set of orbit sums $o(x_{S_1}\dots x_{S_k})$, where $S_1 \subsetneq \cdots \subsetneq S_k$ is a chain, generate $(\mathbb{K}[X]^G, \star)$ as a $(\mathrm{Sym}, \star)$-module. This transfers back to $\mathbb{K}[X]^G$ and Sym.

Note that we may need to consider chains with $S_k = X$; hence the degree bound of $\frac{|X|(|X|+1)}{2}$ instead of $\binom{|X|}{2}$ for permutation groups. For an example where the bound is achieved, consider the group G made of the identity together with all the local bijections of $X = \{1,\dots,n\}$ whose domain is of size at most $|X|-1$; then, $\mathbb{K}[X]^G$ is freely generated as a Sym-module by 1 and the "staircase" monomials $x_1^{d_1},\dots,x_n^{d_n}$ with $1 \leq d_i \leq i$. □

4.2.3. *The Cohen-Macaulay property.* Invariant rings of permutation groups are always Cohen-Macaulay, and in fact free $\mathrm{Sym}(X)$-modules. This follows easily from the fact that the Reynolds operator is a $\mathrm{Sym}(X)$-module morphism. A recent and more involved result is that, for all n, $\mathrm{QSym}(X_n)$ is also a free $\mathrm{Sym}(X_n)$-module [GW03].

As the following example will show, this property does not hold for all permutation groupoids G. Still, $\mathbb{K}[X]^G$ and $(\mathbb{K}[X]^G, \star)$ being finitely generated over $\mathrm{Sym}(X)$, they are Cohen-Macaulay if and only if they are free $\mathrm{Sym}(X)$-modules.

Example 4.13. Let G be the permutation groupoid on $\{1,2,3\}$ of example 4.6 (a), generated by the local bijection $1 \mapsto 2$. Then, G is the restriction of a finite permutation group whose invariant ring is Cohen-Macaulay. However, $\mathbb{K}[X]^G$ itself is not Cohen-Macaulay. Computing the Hilbert series shows right away that this module is not free:

$$\mathcal{H}_{\mathbb{K}[X]^G} = \frac{1}{(1-Z)^3} - \frac{Z}{1-Z} = \frac{1+Z+2Z^2+2Z^3+Z^4-Z^6}{(1-Z)(1-Z^2)(1-Z^3)} \ .$$

To be more explicit, the transitive components of G being $\{1,2\}$ and $\{3\}$, we may replace $\mathrm{Sym}(X)$ by $R = \mathrm{Sym}(x_1,x_2) \otimes \mathrm{Sym}(x_3)$, and view $\mathbb{K}[X]^G$ as a finitely generated R-module. Then, as suggests the Hilbert series,

$$\mathcal{H}_{\mathbb{K}[X]^G} = \frac{1+Z^2+Z^3-Z^4}{(1-Z)^2(1-Z^2)} \ .$$

$\mathbb{K}[X]^G$ is minimally generated as an R-module by $(1, x_1x_3, x_1^2x_2)$, subject to the single relation $x_3.(x_1^2x_2) = (x_1x_2).(x_1x_3)$.

Finally, there is no way of choosing the numerator of the Hilbert series with non-negative coefficients. Indeed, $\mathcal{H}_{\mathbb{K}[X]^G} = \frac{1-Z+2Z^2-Z^3}{(1-Z)^3}$, and the coefficient of highest degree in the product of the numerator by $\frac{(1-Z^{n_1})(1-Z^{n_2})(1-Z^{n_3})}{(1-Z)^3}$ is always -1.

Problem 4.14. Characterize the permutation groupoids G whose invariant rings $\mathbb{K}[X]^G$ (or $(\mathbb{K}[X]^G, \star)$) are Cohen-Macaulay.

The following theorem is a straightforward extension of a theorem of [GS84].

Theorem 4.15. *$(\mathbb{K}[X]^G, \star)$ is a free $\mathrm{Sym}(X)$-module if and only if the incidence matrix between generators and maximal chains is invertible. In particular, for a set F of finely homogeneous invariants whose fine degrees are given by the Hilbert series of $\mathbb{K}[X]^G$, the three following conditions are equivalent: F spans $\mathbb{K}[X]^G$ as a $\mathrm{Sym}(X)$-module, F is a free $\mathrm{Sym}(X)$-family, and F is a $\mathrm{Sym}(X)$-basis of $\mathbb{K}[X]^G$.*

This readily gives us a necessary condition on the number of generators.

Corollary 4.16. *If $(K[X]^G, \star)$ is a free $\mathrm{Sym}(X)$-module, then it is of rank $\frac{|X|!}{|G(X,X)|}$, where $G(X, X)$ is the underlying permutation group of G.*

4.2.4. *SAGBI bases.* SAGBI bases (Subalgebra Analog of a Gröbner Bases for Ideals) were introduced in [KM89, RS90] to develop an elimination theory in subalgebras of polynomial rings. Unlike Gröbner bases, not all subalgebras have a finite SAGBI basis, and it remains a long open problem to characterize those subalgebras which have a one. The following theorem states that, as in the case of permutation groups, invariant rings of permutation groupoids seldom have finite SAGBI bases. The proof follows the short proof given by the second author in [TT04] for permutation groups, with some adaptations. For example $\mathrm{QSym}(X_n)$, represented as a subring of $\mathbb{K}[X]$, has no finite SAGBI basis whenever $n > 1$. In particular, $\mathrm{QSym}(X_2)$ becomes the smallest example of finitely generated algebra which has no finite SAGBI basis (the standard example being the invariant ring of the alternating group A_3). Still, SAGBI bases and SAGBI-Gröbner bases provide a useful device in the computational study of invariant rings of permutation groups [Thi01], and most likely play the same role with permutation groupoids.

Theorem 4.17. *Let G be a permutation groupoid, and $<$ be any admissible term order on $\mathbb{K}[X]$. Then, the invariant ring $\mathbb{K}[X]^G$ has a finite SAGBI basis w.r.t. $<$ if, and only if, G comes from a permutation group generated by reflections (that is transpositions).*

The following proof is a close variant on the short proof given by the second author in [TT04] in the special case of permutation groups. For the sake of readability and completeness, we include it in full here. The key fact is that a submonoid M of $\mathbb{N}^n$ is finitely generated if, and only if, the convex cone $C := \mathbb{R}_+ M$ it spans in $\mathbb{R}_+^n$ is finitely generated (that is C is a *polyhedral cone*). For details, see for example [BG05, Corollary 2.8]. In particular C must be the intersection of finitely many half spaces, and thus closed for the euclidean topology.

Proof. The if-part is easy, a finite SAGBI basis being given by the elementary symmetric polynomials in the variables in each G-transitive components.

Without loss of generality, we may assume $X = \{1, \dots, n\}$ with $x_1 > \dots > x_n$. Let M be the monoid of initial monomials in $\mathbb{K}[X]^G$, seen as a submonoid of $\mathbb{N}^n$, and $C := \mathbb{R}_+ M$ be the convex cone it spans in $\mathbb{R}_+^n$.

At this stage, we cannot give an explicit description of C, but we can construct a convex cone C' which approximates it closely enough for our purposes. By the standard characterization of admissible term orders on $\mathbb{K}[X]$, there exists a family of n linear forms $l = (l_1, \dots, l_n)$ such that $x^d > x^{d'}$ if and only if $l(d) >_{\mathrm{lex}} l(d')$, where we denote by $l(d)$ the n-uple $l_1(d_1, \dots, d_n), \dots, l_n(d_1, \dots, d_n)$. Given two vectors v and v' in $\mathbb{R}_+^n$, we write $v > v'$ if $l(v) >_{\mathrm{lex}} l(v')$. The partial action of G

on monomials extends naturally to a partial action on $\mathbb{R}_+^n$: whenever the support of $v = (v_1, \dots, v_n)$ in contained in the domain of a local bijection $f \in G$, $f.v$ is the vector obtained by permuting the non zero entries of v according to f. Let C' be the subset of all vectors v of $\mathbb{R}_+^n$ such that $v > f.v$ for all $f.v$ in the G-orbit of v. In fact, C' is a convex cone with non empty interior (it contains the n linearly independent vectors $(1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, \dots, 1)$). By construction, M consists of the points of C' with integer coordinates. It follows that $C \subset C'\overline{C}$, where $\overline{C}$ is the topological closure of C.

Assume now that M is finitely generated. Then, C is a closed convex cone, and C and C' simply coincide.

Assume further that G is not generated by transpositions. Then, there exists $a < b$ such that the transposition (a, b) is not in G, while a is in the G-orbit of b. Choose such a pair $a < b$ with b minimal. We claim that there is no transposition (a', b) in G with $a' < b$. Otherwise, a and a' are in the same G-orbit, and by minimality of b, $(a, a') \in G$; thus, $(a, b) = (a, a')(a', b)(a, a') \in G$. Pick $g \in G$ such that $g.b = a$, and for $t \geq 0$, define the vector in $\mathbb{R}_+^n$:

$$u_t := (nt,\ (n-1)t,\ \dots,\ (n-b+2)t,\ n-b+1,\ (n-b)t,\ \dots,\ t,\ 1)\ .$$

Note that $u_1 = (n, \dots, 1)$ is in C, whereas $u_0 = (0, \dots, 0, n-b+1, 0, \dots 0)$ is not in C because $g.u_0 > u_0$.

Take t such that $0 < t \leq 1$. Then, the vector u_t has no zero coefficients, and in particular its G-orbit coincides with its orbit w.r.t. the underlying permutation group $G(X, X)$. Furthermore, the entries of u_t are all distinct, except when $t = \frac{n-b}{n-a'}$ for some $a' < b$, in which case the a'-th and b-th entries are equal. Since $(a', b) \notin G$, the orbit of u_t is of size $|G(X, X)|$, and there exists a unique permutation $f_t \in G(X, X)$ such that $f_t.u_t$ is in C.

Let $t_0 = \inf\{t \geq 0, u_t \in C\}$. If $u_{t_0} \notin C$, then u_{t_0} is in the closure of C, but not in C, a contradiction. Otherwise, $u_{t_0} \in C$, and $t_0 > 0$ because $u_0 \notin C$. For any permutation f, $\{f.u_t, t \geq 0\}$ is a half-line; so, C being convex and closed, $I_f := \{t, f.u_t \in C\}$ is a closed interval $[x_f, y_f]$. For example, $I_{\mathrm{id}} = [t_0, 1] \subsetneq [0, 1]$. Since the interval $[0, 1]$ is the union of all the I_f, there exists $f \neq \mathrm{id}$ such that $t_0 \in I_f$. This contradicts the uniqueness of f_{t_0}. □

4.3. **Stability by derivation.** We denote by ∂_i the derivative w.r.t. the variable x_i, and consider the derivation $D := \sum_{i \in X} \partial_i$ on $\mathbb{K}[X]$.

Proposition 4.18. *Let G be a permutation groupoid. Then $\mathbb{K}[X]^G$ is stable by the derivation D if and only if G comes from a permutation group. On the other hand, $\mathbb{K}[X]^G$ is always stable w.r.t. the action of the* rational Steenrod operators $S_k := \sum_i x_i^{k+1} \partial_i$ *for $k \geq 0$ (see* [HT04] *for details on the rational Steenrod operators).*

Proof. The if part is trivial, since D commutes with the action of the symmetric group $\mathfrak{S}_X$ on $\mathbb{K}[X]$. Similarly, the rational Steenrod operators always stabilize $\mathbb{K}[X]^G$ because they commute with the action of any local bijection on $\mathbb{K}[X]^G$.

Assume now that $\mathbb{K}[X]^G$ is stable by derivation. Let $f : A \mapsto B$ be a local bijection such that $A \subsetneq X$, and take i in $X \backslash A$. We just need to prove that f extends to a local bijection g in G with domain $A \cup \{i\}$. Applying induction, any local bijection in G will then extend to a permutation, as desired.

Take a monomial m whose support is A and whose exponents are all distinct and at least 2, and consider the derivation $p = D(o(mx_i))$ of the orbitsum of the

monomial mx_i in $\mathbb{K}[X]^G$. The monomial m occurs in p; hence, by invariance of p, $f(m)$ also occurs in p, as the derivative of some monomial $g(mx_i)$ in the orbit of mx_i. By the choice of the exponents of m, f and g must coincide on A, while at the same time i belongs to the domain of g. $\square$

Example 4.19. $\mathrm{QSym}(X_2)$ has no graded derivation of degree -1.

References

[BG05] Winfried Bruns and Joseph Gubeladze. Polytopes, rings, and k-theory. February 2005.

[BM04] Rikard Bogvad and Thomas Meyer. On algorithmically checking whether a hilbert series comes from a complete intersection. Research Reports in Mathematics 5, Department of Mathematics, Stockholm University, may 2004.

[Cam90] Peter J. Cameron. *Oligomorphic permutation groups*, volume 152 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

[Cam97] Peter J. Cameron. The algebra of an age. In *Model theory of groups and automorphism groups (Blaubeuren, 1995)*, volume 244 of *London Math. Soc. Lecture Note Ser.*, pages 126–133. Cambridge Univ. Press, Cambridge, 1997.

[FP71] Roland Fraïssé and Maurice Pouzet. Interprétabilité d'une relation pour une chaîne. *C. R. Acad. Sci. Paris Sér. A-B*, 272:A1624–A1627, 1971.

[Fra71] Roland Fraïssé. *Cours de logique mathématique. Tome 1: Relation et formule logique.* Gauthier-Villars Éditeur, Paris, 1971. Deuxième édition revue et modifiée, Collection de "Logique Mathématique", Série A, No. 23.

[Fra00] Roland Fraïssé. *Theory of relations*, volume 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, revised edition, 2000. With an appendix by Norbert Sauer.

[GS84] A. M. Garsia and D. Stanton. Group actions of Stanley - Reisner rings and invariants of permutation groups. *Adv. in Math.*, 51(2):107–201, 1984.

[GW03] A. M. Garsia and N. Wallach. Qsym over Sym is free. *J. Combin. Theory Ser. A*, 104(2):217–263, 2003.

[Hiv04] Florent Hivert. Local actions of the symmetric group and generalisations of quasi-symmetric functions. Preprint, October 2004.

[HM88] I. M. Hodkinson and H. D. Macpherson. Relational structures determined by their finite induced substructures. *J. Symbolic Logic*, 53(1):222–230, 1988.

[HT04] F. Hivert and N. M. Thiéry. Deformation of symmetric functions and the rational Steenrod algebra. In *Invariant theory in all characteristics*, volume 35 of *CRM Proc. Lecture Notes*, pages 91–125. Amer. Math. Soc., Providence, RI, 2004.

[KM89] Deepak Kapur and Klaus Madlener. A completion procedure for computing a canonical basis for a k-subalgebra. In *Computers and mathematics (Cambridge, MA, 1989)*, pages 1–11. Springer, New York, 1989.

[NTT04] Jean-Christophe Novelli, Jean-Yves Thibon, and Nicolas M. Thiéry. Algèbres de Hopf de graphes. *C. R. Math. Acad. Sci. Paris*, 339(9):607–610, 2004.

[Pou76] Maurice Pouzet. Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations. *Math. Z.*, 150(2):117–134, 1976.

[Pou78] Maurice Pouzet. *Sur la théorie des relations*. PhD thesis, Thèse d'état, Université Claude-Bernard, Lyon 1, 1978.

[Pou81] Maurice Pouzet. Application de la notion de relation presque-enchaînable au dénombrement des restrictions finies d'une relation. *Z. Math. Logik Grundlag. Math.*, 27(4):289–332, 1981.

[Pou02] Maurice Pouzet. The profile of relations. Preprint, lecture at Simon Bolivar University, August 2002.

[PR86] M. Pouzet and I. G. Rosenberg. Sperner properties for groups and relations. *European J. Combin.*, 7(4):349–370, 1986.

[RS90] Lorenzo Robbiano and Moss Sweedler. Subalgebra bases. In *Commutative algebra (Salvador, 1988)*, pages 61–87. Springer, Berlin, 1990.

[Thi01] Nicolas M. Thiéry. Computing minimal generating sets of invariant rings of permutation groups with SAGBI-Gröbner basis. In *Discrete models: combinatorics, computation,*

and geometry (Paris, 2001), Discrete Math. Theor. Comput. Sci. Proc., AA, pages 315–328 (electronic). Maison Inform. Math. Discrèt. (MIMD), Paris, 2001.

[TT04] N. M. Thiéry and S. Thomassé. Convex cones and SAGBI bases of permutation invariants. In *Invariant theory in all characteristics*, volume 35 of *CRM Proc. Lecture Notes*, pages 259–263. Amer. Math. Soc., Providence, RI, 2004.

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